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(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FIRST SEMESTER EXAMINATION, DECEMBER 2015

FIRST YEAR [BATCH 2015-18]

MATHEMATICS [Hons]

Paper : I

Date : 14/12/2015

Time : 11 am – 3 pm

Full Marks : 100

[Use a separate Answer Book for each Group]

Group - A

Answer **any five** questions :

1. a) Let $f: X \to Y$ be a map and $B \subseteq Y$. Prove that $f(f^{-1}(B)) \subseteq B$ and the equality holds if f is surjective. [3]

   b) Let ρ be a relation on a set S. Prove that ρ is an equivalence relation if ρ is reflexive and $a\rho b$, $b\rho c$ implies $c\rho a$ for all $a, b, c \in S$. [2]

2. Suppose $\mathbb{Q}' = \mathbb{Q} - \{1\}$. Prove that $\mathbb{Q}'$ forms a commutative group with respect to the composition '$*$' on $\mathbb{Q}'$ defined by : $a * b = a + b - ab \;\forall a, b \in \mathbb{Q}'$. [5]

3. Prove that any finite semigroup in which both cancellation laws hold is a group. [5]

4. Define a binary composition on $G = \{1, 3, 7\}$ to make it a group. Justify your answer. [5]

5. a) Let S_4 be the set of all permutations on a set {1,2,3,4}. Prove that there exists at least one element f in S_4 such that f can not be expressed as g^4 for any $g \in S_4$. [4]

   b) Let $\alpha = (1, 4, 2, 3)$ and $\beta = (1, 3)(2, 4)$ be two permutations in S_4. Compute $\alpha\beta^{-1}$. [1]

6. Prove that every subgroup of a cyclic group is cyclic. [5]

7. a) Let $n \in \mathbb{N}$. Construct a group G and a subgroup H such that $[G : H] = n$. [3]

   b) Show that S_n is nonabelian if n > 2. [2]

8. a) Let (G,$*$) be a group and $a, b \in G$. If $a^2 = e$ and $a * b^2 * a = b^3$ then prove that $b^5 = e$. [3]

   b) Calculate the number of elements of order 2 in a cyclic group of order 10. [2]

Answer **any five** questions :

9. a) Prove that if A be a non-empty set, there is no surjection $\varphi: A \to \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the power set of A. [4]

   b) State the least upper bound axiom. [1]

10. a) Let S be a nonempty subset of $\mathbb{R}$ bounded below. Suppose ℓ is a lower bound of S such that for each natural number n there exists an element x_n in S satisfying $x_n < \ell + 1/n$. Prove that $\ell = \inf S$, where inf S denotes the greatest lower bound of S. [4]

    b) Find sup A and inf A where $A = \{x \in R : x^2 < 1\}$. [1]

11. a) Prove that any superset of any dense set in $\mathbb{R}$ is dense in $\mathbb{R}$. [2]

    b) Prove that the set of all rational numbers is not a closed set. [3]

12. a) Let A be a countable subset of $\mathbb{R}$. Examine whether $A \cup \left\{\bigcap_{n=1}^{\infty}\left[-\frac{1}{n+1}, \frac{1}{n+2}\right]\right\}$ is countable. [3]

    b) Show that every nonempty open set in $\mathbb{R}$ can be expressed as a union of open intervals. [2]

13. a) If $\lim_{n\to\infty} x_n = \ell$ and a > 0 then prove that $\lim_{n\to\infty} a^{x_n} = a^{\ell}$. [3]

b) Give examples of two nonconvergent sequences $\{x_n\}$ and $\{y_n\}$ such that the sequence $\{x_n y_n\}$ is convergent. [2]

14. Prove that the sequence $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$ is a monotone decreasing sequence and convergent. [5]

15. Let D be a subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D and $\ell \in \mathbb{R}$. Then prove that $\lim_{x\to c} f(x) = \ell$ if and only if for every sequence $\{x_n\}$ in D – {c} converging to c, the sequence $\{f(x_n)\}$ converges to ℓ. [5]

16. a) Prove that $\lim_{x\to 0} x\cos\frac{1}{x} = 0$.

b) Let $f : (0,1) \to \mathbb{R}$ be a function such that

$$f(x) = \begin{cases} 1, \text{if } x \text{ is rational} \\ -1, \text{if } x \text{ is irrational} \end{cases}$$

Prove that $\lim_{x\to a} f(x)$ does not exist if $a \in [0,1]$. [5]

Group - B

Answer <u>any five</u> questions :

17. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, then prove that the equation to the third pair of straight lines passing through the points where these lines meet the axes is $ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \frac{4fg}{c}xy = 0$. [5]

18. Find the polar equation of the chord of the conic $\frac{\ell}{r} = 1 - e\cos(\theta + \delta)$ joining the points on the conic with vectorial angles $\alpha - \beta$ and $\alpha + \beta$. [5]

19. Show that the pole of any tangent to the hyperbola xy = c^2 with respect to the circle $x^2 + y^2 = a^2$ lies on concentric and similar hyperbola. [5]

20. Reduce the equation : $3(x^2 + y^2) + 2xy = 4\sqrt{2}(x + y)$ to its canonical form. Name the conic and find the eccentricity of the conic. [3+1+1]

21. Show that the tangents at the extremities of conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersect on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$. [5]

22. Let D, E, F be respectively the midpoints of the sides BC, CA, AB of the triangle ABC. Show that the vector area of ΔABC is four times that of ΔDEF. [5]

23. Prove the formula : $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 0$ and use it to show that $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$ for any two acute angles A and B. [2+3]

24. Find the shortest distance between the two lines through A (6, 2, 2) and C (–4, 0, –1) and parallel to the vectors $\hat{i} - 2\hat{j} + 2\hat{k}$ and $3\hat{i} - 2\hat{j} - 2\hat{k}$ respectively. Find the points where the lines meet the shortest distance line. [3+1+1]

Answer <u>any five</u> questions :

25. Prove that if $y=\dfrac{\ell x+m}{px+q}$ where ℓ, m, p, q are arbitrary constants then it satisfies the differential equation $2\dfrac{dy}{dx}\cdot\dfrac{d^3y}{dx^3}=3\left(\dfrac{d^2y}{dx^2}\right)^2$ and when $\ell+q=0$ then it satisfies $(y-x)\dfrac{d^2y}{dx^2}=2\left(1+\dfrac{dy}{dx}\right)\dfrac{dy}{dx}$. [5]

26. Show that the equation $(P+Qx)\dfrac{dy}{dx}=R+Qy$, where P, R are homogeneous functions in x and y of degree n and Q is a homogeneous function in x and y of degree m, can be solved by the substitution y = vx. [5]

27. Reducing the differential equation $x^2p^2+py(2x+y)+y^2=0$ to Clairaut's form by the substitution y = u, xy = v, solve it and prove that y + 4x = 0 is a singular solution. [5]

28. Prove that the system of confocal conics $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$ is self orthogonal. [5]

29. Solve, by the method of variation of parameters, the equation $\dfrac{d^2y}{dx^2}-y=\dfrac{2}{1+e^x}$. [5]

30. Using the method of undetermined coefficients, solve the equation $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+3y=x^3+\sin x$. [5]

31. State the sufficient condition of linear independence of the solutions of a general linear differential equation of order n. Show that the equation $x^3\dfrac{d^3y}{dx^3}-6x\dfrac{dy}{dx}+12y=0$ has three linearly independent solutions of the form $y = x^r$. [1+4]

32. Solve : $x^3\dfrac{d^3y}{dx^3}+2x^2\dfrac{d^2y}{dx^2}+2y=10\left(x+\dfrac{1}{x}\right)$. [5]

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